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Review of A. Hatcher & J. Wagoner's paper
'Pseudo-isotopies of compact manifolds' (Part II)

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Let M^n be a connected compact smooth manifold and $\mathcal{P} = \mathcal{P}(M,\partial M)$ be the pseudo-isotopy space of M.

The aim of the paper [1] by A. Hatcher & J. Wagoner is the computation of $\pi_0(\mathcal{P})$. As "Part II" of the introduction, we shall explain how the first obstruction $\Sigma: \pi_0(\mathcal{P}) \longrightarrow Wh_2(\pi_1 M)$ is constructed.

$\pi_0(\mathcal{P})$ is replaced by $\pi_1(\mathcal{F},\mathcal{E}:p)$, so we start from $\pi_1(\mathcal{F},\mathcal{E}:p)$.

<u>Theorem 1.</u> There is a surjection $\Sigma: \pi_1(\mathcal{F},\mathcal{E}:p) \longrightarrow Wh_2(\pi_1 M)$.

Our target group $Wh_2(\pi_1 M)$ has another presentation. For simplicity, let $\Lambda = Z[\pi_1 M]$ and $G = \pi_1 M$.

<u>Proposition 0.</u> $Wh_2(\pi_1 M) \cong U(\Lambda)/U(\pm G)$,

where $U(\Lambda) = \{x \in St(\Lambda) \mid \pi(x) = (a_{pq}),\ \pi: St(\Lambda) \longrightarrow E(\Lambda)$

(1) $a_{pq} = 0$ if $q<p$

(2) $a_{pp} = \pm g_p$ for some $g_p \in G\ \}$,

(This is a subgroup of $St(\Lambda)$.)

$U(\pm G)$ = the subgroup of $U(\Lambda)$, generated

by $\left\{\begin{array}{l} w_{pq}(\pm g)\cdot w_{pq}(-1) \quad g \in G \\ x_{pq}(\lambda) \quad \text{with } p<q, \quad \lambda \in \Lambda. \end{array}\right.$

$[f_t] \in \pi_1(\mathcal{F},\mathcal{E}:p)$ be a one parameter family of functions $f_t: M\times I \longrightarrow I$, $t \in [0,1]$, where $f_0 = p$:the standard projection and $f_1 = f$ are in $\mathcal{E}$. We make the following deformations, keeping both ends fixed and without changing the homotopy class of $[f_t]$. The geometrical details are complicated so we give only a rough sketch.

<u>1st step.</u> By the stratification theory of the function space $\mathcal{F}$, we can approximate the one parameter family by a generic family. Here the generic family consists of the Morse functions f_t except for finite t, and for finite t, f_t is a function with a birth (or a death) point and some non-degenerate critical points.

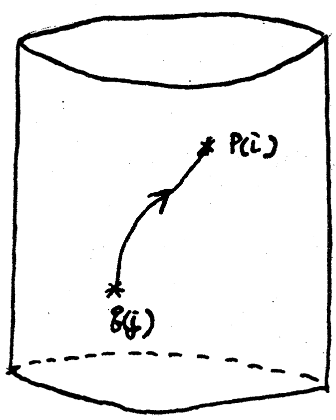
<u>2nd step.</u> Choose a one parameter family $\{\eta_t\}$ of gradient like vecter fields for $\{f_t\}$.

Then we can consider the trajectories.

Let p and q be the two critical points of f_t. If there are no trajectories leading up from p to q (or q to p), we say p and q are independent.

<u>Proposition 1.</u> If p is a birth (or a death) point, then $\{\eta_t\}$ can be deformed, for which p is independent of all the other critical points.

Next, let p and q be the two non-degenerate critical points of index i and j. The trajectory from q to p is called the i/j-intersection. Using the general position methods, we deform the path $\{\eta_t, f_t\}$, then by the dimensional reason, there are no i/j-intersections for $i<j$, and there are only a finite number of i/i-intersections, they are important for us and they are also called "gradient crossings" or "handle additions".



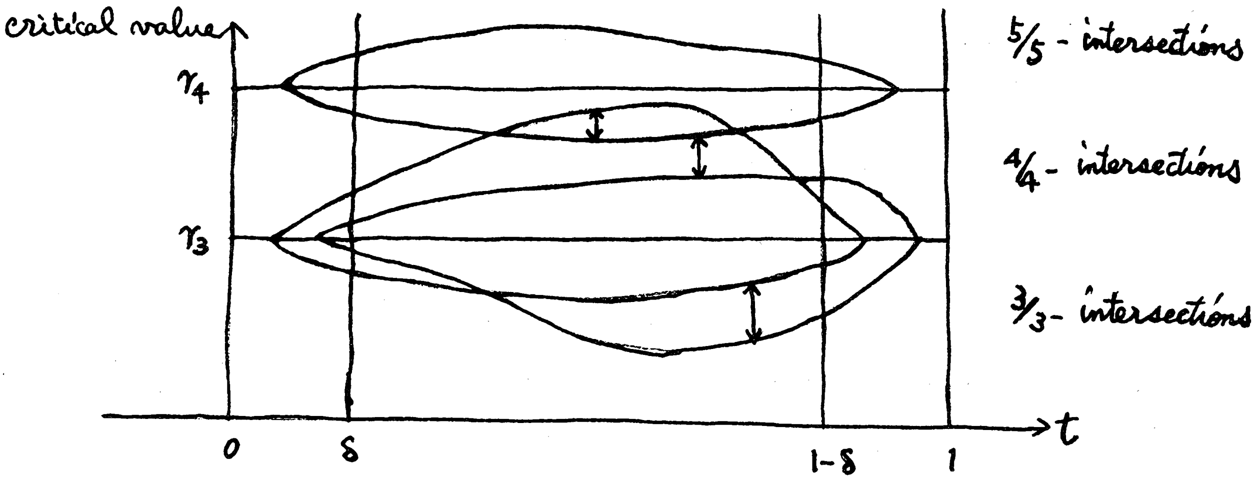
$M \times I$

For fixed t, there are a finite number of $i/(i-1)$-intersections, they are the incidence points.

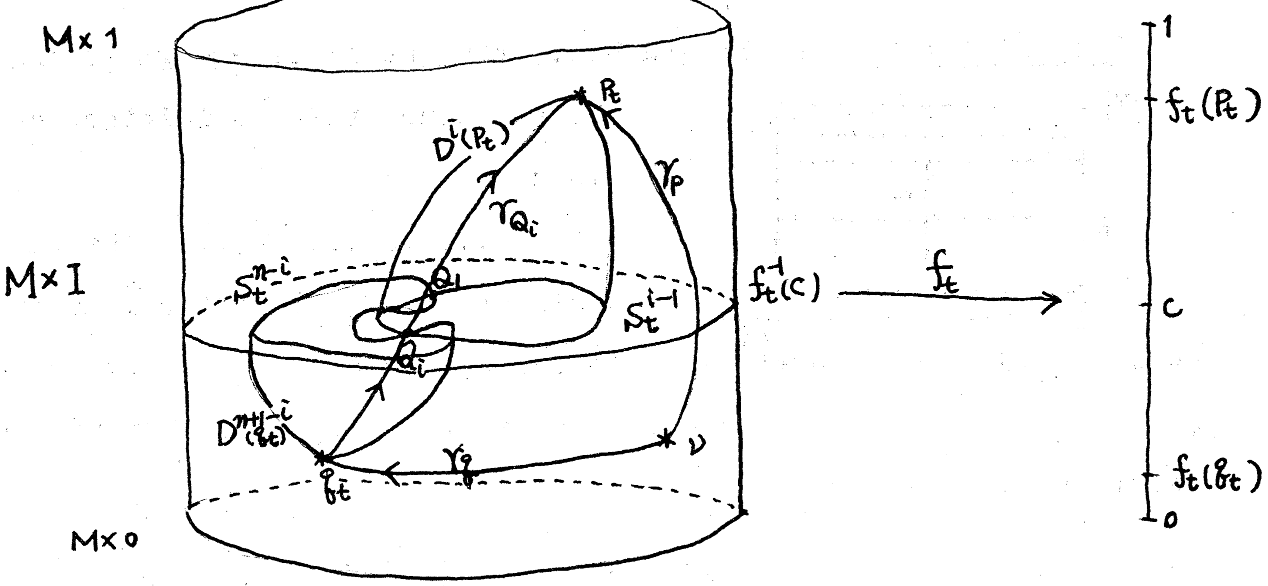
<u>3rd step.</u> By the independent trajectory principle, we make $\{\eta_t, f_t\}$ to be an ordered family. That is, for $0<r_0<r_1<\ldots r_n<1$, if p is a degenerate critical point of f_t, of index i, then $f_t(p) = r_i$ and if p is a non-degenerate critical point of f_t, of index i, then $f_t(p) \in [r_{i-1}, r_i]$.

For small $\delta>0$, we make more deformations, so that all the birth points occur in $[0,\delta]$, and all the death points occur in $[1-\delta,1]$, and there are no i/i-intersections in $[0,\delta] \cup [1-\delta,1]$.

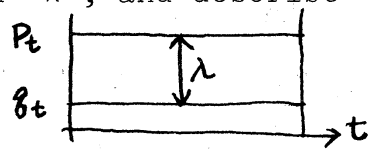
For example, the graphic will be:



Let us explain more about the i/i-intersections. For finite t, there are finitely many i/i-intersections from q_t to p_t. Let $S_t^{n-i} \cap S_t^{i-1} = \{Q_1, Q_2, \ldots, Q_k\}$, where S_t^{n-i} is the unstable sphere of q_t and S_t^{i-1} is the stable sphere of p_t, the both are in the middle level surface $f_t^{-1}(c)$, for $f_t(q_t)<c<f_t(p_t)$. Let ν be the base point of $M \times I$, and choose the base paths γ_p and γ_q, and γ_{Q_i} be the i/i-intersection. Then the composition $\gamma_p * \gamma_{Q_i}^{-1} * \gamma_q^{-1}$ decides an element in $G = \pi_1 M$.



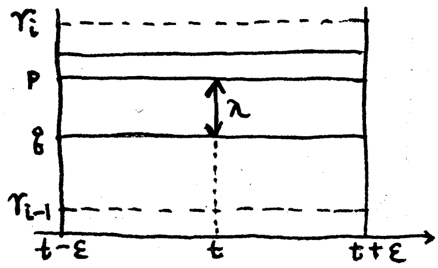
If we give the orientations to $W(p_t)$ and $W(q_t)$, there exists an intersection number $\varepsilon_{Q_i} \in \{\pm 1\}$, and $\Sigma_{i=1}^{k} \varepsilon_{Q_i}\ [\gamma_p * \gamma_{Q_i}^{-1} * \gamma_q^{-1}] = \lambda \in \Lambda$. This $\lambda \in \Lambda$ is the algebraic intersection number. We call this set of i/i-intersections "i/i-intersection λ", and describe on the graphic by a vertical arrow as follows:



If we give p_t & q_t, the indices of Steinberg group, for example, p and q, then a Steinberg symbol $x_{pq}(\lambda) \in St(\Lambda)$ corresponds to each i/i-intersection λ.

Next, we give the algebraic property of the i/i-intersections. For (η_t, f_t), where f_t is a Morse function, there is a chain complex, which is defined in the following way.
Let $(V_i;\ \partial_- V_i,\ \partial_+ V_i) = (f_t^{-1}([r_{i-1}, r_i]),\ f_t^{-1}(r_{i-1}),\ f_t^{-1}(r_i))$. Let $p\colon \widetilde{M\times I} \longrightarrow M\times I$ be the universal cover of $M\times I$ and for any subset $A \subset M\times I$, let $p^{-1}(A) = \overline{A}$. Choose paths from a fixed base point to each critical point and orient the stable manifold of each critical point as in the s-cobordism theory. We have $(C_*,\ \partial_*)$, where $C_i(f_t) = H_i(\overline{V_i},\ \overline{\partial_- V_i})$ is a free Λ-module, whose basis are determined by the liftings of the stable disks of index i critical points of f_t.

Proposition 2. In the graphic, $\varepsilon>0$ be small enough so that there are no other i/i-intersections.



Let $\varepsilon_1,\ldots,\varepsilon_p,\ldots,\varepsilon_q,\ldots$ be the basis of $C_i(f_{t-\varepsilon})$ determined by the stable disks of index i critical points of $f_{t-\varepsilon}$, then $\varepsilon_1,\ldots,\varepsilon_p+\lambda\varepsilon_q,\ldots,\varepsilon_q,\ldots$ are the basis of $C_i(f_{t+\varepsilon})$ determined in a similar way.

So there is a transformation of basis $C_i(f_{t-\varepsilon}) \longleftarrow C_i(f_{t+\varepsilon})$, expressed by the elementary matrix $\pi(x_{pq}(\lambda)) = e_{pq}(\lambda)$.

To treat everything at once, we introduce the standard complex (ω,σ), which is defined in the following way.

For $i>0$, let C_i be the free left Λ-module over $\{b_i^\alpha, z_i^\beta\}_{\alpha,\beta\in Z}$, and C_0 be the free left Λ-module over $\{z_0^\beta\}_{\beta\in Z}$. We call b_i^α "the boundary indices" and z_i^β "the cycle indices".

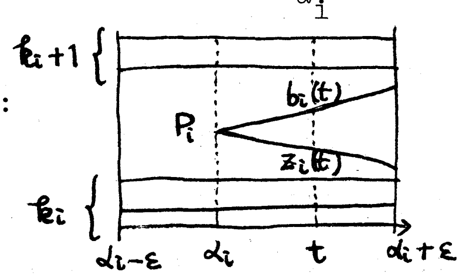
Define the boundary operator and the contraction operator by $\omega = \{\ \omega_i\colon C_i \longrightarrow C_{i-1}\ \}$ $\sigma = \{\ \sigma_i\colon C_i \longrightarrow C_{i+1}\ \}$

$$\begin{cases} \omega_i(b_i^\alpha) = z_{i-1}^\alpha \\ \omega_i(z_i^\beta) = 0 \end{cases} \qquad \begin{cases} \sigma_i(b_i^\alpha) = 0 \\ \sigma_i(z_i^\beta) = b_{i+1}^\beta . \end{cases}$$

In the graphic, $\{p_1,p_2,\ldots,p_m\}$ be the set of all birth points of $\{\eta_t,f_t\}$ such that p_i is a birth point of f_{α_i}, of index k_i, where $\alpha_1<\alpha_2<\ldots<\alpha_m$.

The graphic near time $t = \alpha_i$ looks like:

for small $\varepsilon>0$.



Here $b_i(t)$ and $z_i(t)$ are the couple of non-degenerate critical points,born at p_i.

Choose (a) a base path γ_i from ν to p_i (ν is the base point in $M\times I$.)

(b) the orientations of $W(b_i(t))$ and $W(z_i(t))$ so that $\partial_i(b_i(t)) = +z_i(t)$

(c) for $b_i(t)$, some boundary index $b^{\alpha}_{k_i+1}$ and for $z_i(t)$ the corresponding cycle index $z^{\alpha}_{k_i}$.

Then each i/i-intersection λ has a symbol $x_{pq}(\lambda)\in St_i(\Lambda)$, $p,q\in\{b_i^{\alpha},z_i^{\beta}\}_{\alpha,\beta\in Z}$ $\lambda\in\Lambda$.

For each i, in the graphic read the Steinberg symbols from left to right and multiply and write it down by $x_i\in St_i(\Lambda)$.

The multi-Steinberg word is defined by $x = (x_0,x_1,\dots,x_i,\dots)$ $\in \bigoplus_i St_i(\Lambda)$.

f_δ and $f_{1-\delta}$ are the Morse functions, so we have the chain complexes $C_*(f_\delta)$ and $C_*(f_{1-\delta})$ and the chain transformation between them, because of Proposition 2.

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] \\
C_i(f_\delta) \arrow[d, "\omega_i"] & C_i(f_{1-\delta}) \arrow[l, "\pi(x_i)"'] \arrow[d, "\partial_i"] \\
C_{i-1}(f_\delta) \arrow[u, bend left, "\sigma_{i-1}"] \arrow[d] & C_{i-1}(f_{1-\delta}) \arrow[l, "\pi(x_{i-1})"] \arrow[u, bend left, "\delta_{i-1}"] \arrow[d] \\
{} & {}
\end{tikzcd}$$

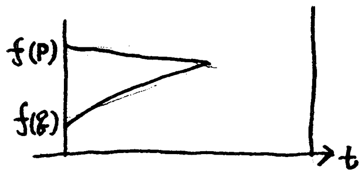
$$\{\ \partial_i = \pi(x_i)\omega_i\pi(x_{i-1})^{-1} = x\omega_i\ \} = x\omega$$

$$\{\ \delta_{i-1} = \pi(x_{i-1})\sigma_{i-1}\pi(x_i)^{-1} = x\sigma_{i-1}\} = x\sigma.$$

After the above choice, we can regard $C_*(f_\delta) = (\omega,\sigma)$, and $C_*(f_{1-\delta}) = x(\omega,\sigma)$. Here is the operation of $x\in\bigoplus_i St_i(\Lambda)$ on (ω,σ).

As $f\in E$ (i.e. f_1 has no critical points), all the critical points of $f_{1-\delta}$ will be cancelled in some death points.

<u>Proposition 3.</u> In the following graphic, p and q can be cancelled, if and only if, $\partial(\mathrm{base}(p)) = \pm g(\mathrm{base}(q))$ for some $g \in G$.



Then, $(x\omega_i)(\text{basis element}) = \begin{cases} \pm g(\text{basis element}) \\ \text{or} \\ 0 \end{cases}$

As $x(\omega,\sigma)$ is almost the standard complex, by the permutation of the indices of each Steinberg group $St_i(\Lambda)$, which is realized by some element $u = (u_0,\ldots,u_i,\ldots) \in \bigoplus_i W_i(\pm G) \subset \bigoplus_i St_i(\Lambda)$, we have the following formula

$$(*)\quad \begin{cases} (ux)\omega_i(b_i^\alpha) = \pm g z_{i-1}^\alpha \\ (ux)\omega_i(z_i^\beta) = 0. \end{cases}$$

Like the Whitehead torsion of the chain complex, $(ux)\omega_{ev} + (ux)\sigma_{ev} : \bigoplus_{i \geqq 0} C_{2i} \longrightarrow \bigoplus_{i \geqq 0} C_{2i+1}$ is an isomorphism and expressed by the matrix

$$\begin{array}{cc} & \begin{array}{cccc} \quad C_1 \quad & \quad C_3 \quad & \quad C_5 \quad & \\ b_1^\alpha \ z_1^\beta & b_3^\alpha \ z_3^\beta & b_5^\alpha \ z_5^\beta & \cdots \end{array} \\ \begin{array}{cc} C_0 & z_0^\beta \\ C_2 & b_2^\alpha \ z_2^\beta \\ C_4 & b_4^\alpha \ z_4^\beta \\ \vdots & \vdots \end{array} & \left(\begin{array}{c|c|c|c} \delta_0 & 0 & 0 & \\ \hline \partial_2 & \delta_2 & 0 & \\ \hline 0 & \partial_4 & \delta_4 & \\ \hline 0 & 0 & & \end{array}\right) \end{array}$$

This matrix is the desired upper trianguler matrix in $\pi(U(\Lambda))$ because of the formula (*) and the contraction formula induced by (*). This matrix is $\pi((\Pi_{i\geqq 0}u_{2i}\cdot x_{2i})(\Pi_{i\geqq 0}x_{2i+1}^{-1}\cdot u_{2i+1}^{-1}))$, where $\pi\colon \oplus St_i(\Lambda) \longrightarrow \oplus E_i(\Lambda)$.

We define $\Sigma\colon \Pi_1(F,E:p) \longrightarrow Wh_2(\pi_1 M)$ by
$\Sigma([f_t]) = (\Pi_{i\geqq 0}u_{2i}\cdot x_{2i})(\Pi_{i\geqq 0}x_{2i+1}^{-1}\cdot u_{2i+1}^{-1})$ mod $U(\pm G)$.

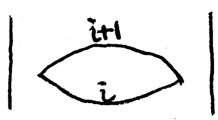
Our first obstruction $\Sigma\colon \pi_0(P) \longrightarrow Wh_2(\pi_1 M)$ is defined by the formula $\Sigma([g]) = \Sigma([\text{path from } p \text{ to } p\circ g])$ for $g \in P(M)$.

The proof of well-definedness is not easy. We have to consider the two-parameter families, and their generic families have some codimension 2 singulalities (dovetail points).

To show the Σ is surjective, for an element z in $Wh_2(\pi_1 M)$, represented by $\Pi x_{pq}(\lambda) \in K_2(\Lambda)$, we construct a path from p to f, where $f \in E$. The constructions are realized by the embeddings of the standard path models. This is done for dim $M \geqq 5$.

The first obstruction describes the i/i-intersections, then the kernel of Σ consists of the one parameter families without i/i-intersections.

Let $\mathcal{D} = \{[f] \in \pi_0(E) \cong \pi_0(P) \mid$ path from p to f has a graphic like:



$\}$.

<u>Proposition 4.</u> $\mathcal{D}$ is a subgroup of $\pi_0(E)$, for dim $M^n \geqq 4$.

<u>Theorem 2.</u> $\mathrm{Ker}\Sigma = \mathcal{D}$, for dim $M^n \geqq 5$.

The birth and death points are crucial in this theory, but they are tame and easy to treat.

A. Hatcher has defined in [2], the 2nd obstruction for $\pi_0(\mathcal{P})$, $\theta\colon \pi_0(\mathcal{P}) \longrightarrow Wh_1(\pi_1 M\colon Z_2 \times \pi_2 M)$, which describes the kernel $\mathcal{D}$. But it has mistakes, those problems are solved by K. Igusa in [7].

References

[1] A. Hatcher and J. Wagoner: Pseudo-isotopies of compact manifolds, astérisque, 6, 1973, Soc. Math. de France, Paris.

[2] A. Hatcher: The second obstruction for pseudo-isotopies, astérisque, 6, 1973, Soc. Math. de France, Paris.

[3] J. Cerf: La stratification naturelle des espace de fonctions différentiables réelles et le théorèm de la pseudo-isotopie, Publ. Math. I.H.E.S. 39(1970).

[4] J. Milnor: Lecture on the h-cobordism theorem, Princeton Mathematical Notes (1965).

[5] J. Milnor: Introduction to Algebraic K-theory, Annals Study #72, Princeton University Press.

[6] M. Kervaire: Le Théorème de Barden-Mazur-Stallings, Comm. Math. Helv. 40(1965), p.31-42.

[7] K. Igusa: What happens to Hatcher and Wagoner's formula for $\pi_0(C(M))$, when the first postnikov invariant of M is nontrivial ?, Springer Lecture Note, 1046.